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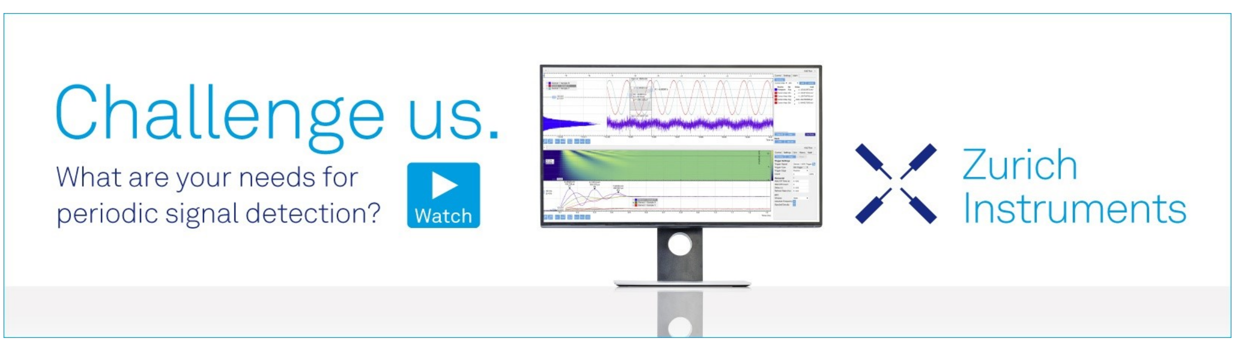

An Approximate Solution of First Derivatives of the Mixed Boundary Value Problem for Laplace's Equation on a Rectangle

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Abstract. In a rectangular domain, we discuss about an approximation of the first order derivatives for the solution of the mixed boundary value problem. The boundary values on the sides of the rectangle are supposed to have the second order derivatives satisfying the Hölder condition. Under these conditions for the approximate values of the first derivatives of the solution of mixed boundary problem on a square grid, as the solution of the constructed difference scheme a uniform error estimation of order $O(h)$ (h is the grid size) is obtained. Numerical experiments are illustrated to support the theoretical results.

INTRODUCTION

Laplace equation is an important equation with many applications in engineering fields. The Laplace equation has numerical solutions that have been studied along with many boundary conditions, the most applicable of which is the mixed boundary condition problem. Many of these studies have used classical operators (see [1] and [2]). As well as the solution of the Laplace equation, the derivative of its solution has many applications (see [3]) and many of these solutions were obtained using classical operators (see [4]).

In this study, the mixed boundary value problem of Laplace equation is considered by using 5-point finite difference scheme on a rectangle. A second order finite-difference approximation on square grids is used to obtain numerical solution. The uniform convergence for the approximate solution at the rate of $O(h^2)$ and for the first derivative at the rate of $O(h)$ is proved when the exact solution u belongs to $\widetilde{C}^{2,\lambda}$.

FINITE DIFFERENCE APPROXIMATION

Let $\Pi = \{(x,y) : 0 < x < a, 0 < y < b\}$ be rectangle, a/b be rational, γ_j, $j = 1,2,3,4$, be the sides, including (excluding) the ends, enumerated counterclockwise starting from the side which location on the $x-$axis ($\gamma_0 \equiv \gamma_4, \gamma_1 \equiv \gamma_5$). Denote by s the arc length, measured along γ, and by s_j the value of s at the beginning of γ_j and by $\gamma = \bigcup_{j=1}^{4} \gamma_j$, the boundary of Π, by v_j a parameter taking the values 0 or 1, and $\bar{v}_j = 1 - v_j$.

We consider the following boundary value problem

$$\Delta u = 0 \text{ on } \Pi, \tag{1}$$

$$v_j u + \bar{v}_j u_n^{(1)} = v_j \varphi_j + \bar{v}_j \psi_j \text{ on } \gamma_j,\ j = 1,2,3,4, \tag{2}$$

where $u_n^{(1)}$ is the derivative along the inner normal, φ_j and ψ_j are given functions at the arc length taken along γ,

$$1 \le \sum_{j=1}^{4} v_j \le 4,\ v_1 = 1. \tag{3}$$

Definition 1 *We say that the solution u of the problem (1), (2) belongs to $\widetilde{C}^{k,\lambda}(\overline{\Pi})$, if*

$$v_j \varphi_j + \bar{v}_j \psi_j \in C^{k,\lambda}(\gamma_j),\ 0 < \lambda < 1,\ j = 1,2,3,4 \tag{4}$$

and at the vertices $A_j = \gamma_{j-1} \cap \gamma_j$ the conjugation conditions

$$v_j \varphi_j^{2q+\delta_{\tau-2}} + \bar{v}_j \psi_j^{2q+\delta_\tau} = (-1)^{q+\delta_\tau+\delta_{\tau-1}} (v_{j-1} \varphi_{j-1}^{2q+\delta_{\tau-1}} + \bar{v}_{j-1} \psi_{j-1}^{2q+\delta_\tau}) \tag{5}$$

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are satisfied, except may be the case when $q = k/2$ *for* $\tau = 3$, *where* $\tau = \nu_{j-1} + 2\nu_j$, $\delta_w = 1$ *for* $w = 0$; $\delta_w = 0$ *for* $w \neq 0$, $q = 0, 1, ..., Q$, $Q = [(k - \delta_{\tau-1} - \delta_{\tau-2})/2] - \delta_\tau$.

Let $h > 0$, with $a/h \geq 2$, $b/h \geq 2$ be integers. We assign to Π^h a square net on Π, with step h, obtained by the lines $x, y = 0, h, 2h, ...$; γ_j^h be a set of nodes on the interior of γ_j, and let

$$\dot{\gamma}_j^h = \gamma_j \cap \gamma_{j+1},\ \gamma^h = \cup(\gamma_j^h \cup \dot{\gamma}_j^h),\ \overline{\Pi}^h = \Pi^h \cup \gamma^h.$$

Let the operators A, $\boldsymbol{K}$ and $\dot{\boldsymbol{K}}$ be defined as follows:

$$Au(x,y) = \frac{u(x+h,y) + u(x,y+h) + u(x-h,y) + u(x,y-h)}{4} \text{ on } \Pi^h, \tag{6}$$

$$\boldsymbol{K}u(x,y) = \frac{1}{2}u(x + h\sin\frac{j\pi}{2}, y - h\cos\frac{j\pi}{2}) + \frac{1}{4}\sum_{k=0}^{1} u(x + (-1)^k h\cos\frac{j\pi}{2}, y + (-1)^k h\sin\frac{j\pi}{2}) \text{ on } \gamma_j^h,\ j = 1,2,3,4, \tag{7}$$

and

$$\dot{\boldsymbol{K}}u(x,y) = \frac{1}{2}\sum_{k=0}^{1} u(x + h\sin\frac{(j+k)\pi}{2}, y - h\cos\frac{(j+k)\pi}{2}) \text{ on } \dot{\gamma}_j^h,\ j = 1,2,3,4. \tag{8}$$

We consider the classical 5-point finite difference approximations of the problem (1):

$$u_h = Au_h \text{ on } \Pi^h, \tag{9}$$

$$u_h = \nu_j\varphi_j + \bar{\nu}_j(\boldsymbol{K}u_h - \frac{h}{2}\psi_j) \text{ on } \gamma_j^h,\ j = 1,2,3,4, \tag{10}$$

$$u_h = \nu_j\varphi_j + \bar{\nu}_j\nu_{j+1}\varphi_{j+1} + \bar{\nu}_j\bar{\nu}_{j+1}(\dot{\boldsymbol{K}}u_h - \frac{h}{2}(\psi_j + \psi_{j+1})) \text{ on } \dot{\gamma}_j^h,\ j = 1,2,3,4. \tag{11}$$

The system of finite difference equations (9)-(11) which has nonnegative coefficients, with the condition (3) is uniquely solvable.

Theorem 1 *Let* u *the solution of problem (1), (2). If* $u \in \widetilde{C}^{2,\lambda}(\overline{\Pi})$ *the condition (3) holds, then*

$$\max_{\overline{\Pi}^h} |u_h - u| \leq ch^2,$$

where u_h *is the solution of the system (9)-(11).*

The proof of Theorem 1 follows from the Theorem 1 in [1].

APPROXIMATE OF THE FIRST DERIVATIVES

Let u be a solution of problem (1), (2). Let $v = \frac{\partial u}{\partial x}$ and let $\Phi_j = \frac{\partial u}{\partial x}$ on γ_j, $j = 1,2,3,4$, and consider the boundary value problem:

$$\Delta v = 0 \text{ on } \Pi,\ v = \Phi_j \text{ on } \gamma_j,\ j = 1,2,3,4. \tag{12}$$

We define the sets

$$\gamma_{(2^i-1)}^{h+} = \left\{0 \leq x \leq \frac{a}{2},\ y = b(i-1)\right\} \cap \gamma_{(2^i-1)}^h,\ i = 1,2 \tag{13}$$

and

$$\gamma^{h-}_{(2^i-1)} = \left\{\frac{a}{2}+h \le x \le a,\ y = b(i-1)\right\} \cap \gamma^h_{(2^i-1)},\ i = 1,2. \tag{14}$$

We define the following operator $\Phi_{ph},\ p = 1,2,3,4,$

$$\Phi_{(2^i-1)h}(u_h) = v_3\frac{\partial u(x,b(i-1))}{\partial x} + \bar{v}_3\frac{1}{h}[-u_h(x,b(i-1)) + u_h(x+h,b(i-1))] \text{ on } \gamma^{h+}_{(2^i-1)}, \tag{15}$$

$$\Phi_{(2^i-1)h}(u_h) = v_3\frac{\partial u(x,b(i-1))}{\partial x} + \bar{v}_3\frac{1}{h}[u_h(x,b(i-1)) - u_h(x-h,b(i-1))] \text{ on } \gamma^{h-}_{(2^i-1)}, \tag{16}$$

$$\Phi_{(2i)h}(u_h) = \bar{v}_{(2i)}\psi_{(2i)} + v_{(2i)}\frac{(-1)^{i+1}}{h}[\varphi_{(2i)}((2-i)a) - u_h((2-i)a + (-1)^i h, y)] \text{ on } \gamma^h_{(2i)}, \tag{17}$$

where $i = 1,2$ and u_h is the solution of the finite difference problem (9)-(11).
Let v_h be the solution of the following finite difference problem

$$v_h = Av_h \text{ on } \Pi_h,\ v_h = \Phi_{jh} \text{ on } \gamma^h_j,\ j = 1,2,3,4, \tag{18}$$

where Φ_{jh} , $j = 1,2,3,4$, are defined by (15)-(17).

Theorem 2 *The following estimation is true*

$$\max_{(x,y)\in\Pi^h}\left|v_h - \frac{\partial u}{\partial x}\right| \le ch,$$

where u is the solution of the problem (1), v_h is the solution of the finite difference problem (18).

Remark 1 *We have investigated approximations of the first derivative $\frac{\partial u}{\partial x}$. The same results are obtained for the derivative $\frac{\partial u}{\partial y}$, by using the same order forward and backward formulae in the corresponding sides of the rectangular domain.*

NUMERICAL EXAMPLES

Example 1 *Let $\Pi = \{(x,y) : 0 < x,y < 1\}$, and let γ be the boundary of Π. We consider the following problem :*

$$\Delta u = 0 \text{ on } \Pi,\ u = \varphi_j(x,y) \text{ on } \gamma_j,\ j = 1,2,3,\ u^{(1)} = \frac{\partial u(0,y)}{\partial x} = \psi_4(y) \text{ on } \gamma_4,$$

where

$$\varphi_j(x,y) = (x^2+y^2)^{\frac{61}{60}}\cos\left(\frac{61}{30}\arctan\left(\frac{y}{x}\right)\right) \text{ on } \gamma_j,\ j = 1,2,3,$$

and

$$\psi_4(y) = \frac{61}{30}y^{\frac{31}{30}}\sin\left(\frac{61\pi}{60}\right) \text{ on } \gamma_4.$$

Table 1 shows that the order of the solution of the problem given in Example 1 is $O(h^2)$ when the given Neumann condition on the left side and order of the first derivative $O(h)$ when used the first order backward numerical differentiation formula on the right side.

Example 2 *Let $\Pi = \{(x,y) : 0 < x,y < 1\}$, and let γ be the boundary of Π. We consider the following problem :*

$$\Delta u = 0 \text{ on } \Pi,\ u = \varphi(x,y) \text{ on } \gamma_j,\ j = 1,2,3,$$

$$u^{(1)} = \frac{\partial u(x,1)}{\partial y} = \psi_3(y) \text{ on } \gamma_3 \text{ and } u^{(1)} = \frac{\partial u(0,y)}{\partial x} = \psi_4(y) \text{ on } \gamma_4,$$

TABLE 1. The approximate results for the solution and first derivative when $\varphi \in C^{2,\frac{1}{30}}$

h	$\|u_h - u\|$	E_u^m	$\|v_h - v\|$	E_v^m
$\frac{1}{4}$	$3.224E-04$	3.631	$2.641E-01$	1.994
$\frac{1}{8}$	$8.878E-05$	3.879	$1.324E-01$	1.997
$\frac{1}{16}$	$2.289E-05$	3.988	$6.631E-02$	1.999
$\frac{1}{32}$	$5.739E-06$	4.032	$3.318E-02$	1.999
$\frac{1}{64}$	$1.423E-06$	4.040	$1.659E-02$	2.000
$\frac{1}{128}$	$3.523E-07$		$8.299E-03$	

where

$$\varphi_j(x,y) = (x^2+y^2)^{\frac{61}{60}} \cos\left(\frac{61}{30} \arctan\left(\frac{y}{x}\right)\right) \ on\ \gamma_j,\ j = 1,2,3,$$

and

$$\psi_3(y) = -\frac{61}{30}\sqrt[60]{x^2+1}\left[x \sin(\frac{61}{30}\arctan(\frac{1}{x}) - \cos(\frac{61}{30}\arctan(\frac{1}{x}))\right] \ on\ \gamma_3,$$

$$\psi_4(y) = \frac{61}{30} y^{\frac{31}{30}} \sin\left(\frac{61\pi}{60}\right) \ on\ \gamma_4.$$

TABLE 2. The approximate results for the solution and first derivative when $\varphi \in C^{2,\frac{1}{30}}$

h	$\|u_h - u\|$	E_u^m	$\|v_h - v\|$	E_v^m
$\frac{1}{4}$	$7.404E-04$	3.999	$2.663E-01$	2.002
$\frac{1}{8}$	$1.851E-04$	4.007	$1.330E-01$	2.001
$\frac{1}{16}$	$4.621E-05$	4.006	$6.647E-02$	2.001
$\frac{1}{32}$	$1.153E-05$	4.004	$3.332E-02$	2.000
$\frac{1}{64}$	$2.881E-06$	4.002	$1.661E-02$	2.000
$\frac{1}{128}$	$7.199E-07$		$8.301E-03$	

Table 2 shows that the order of the solution of the problem given in Example 2 is $O(h^2)$ when the given Neumann condition on the left side and at the top sides and order of the first derivative $O(h)$ when used the first order forward and backward numerical differentiation formula on the right and up sides.

The results which illustrated in Table 1 and Table 2 are the numerical justification of Theorem 1 and Theorem 2.

In Table 1 and Table 2 we have used the following notations:

$\|U_h - U\|_{\overline{\Pi}^h} = \max_{\overline{\Pi}^h} |U_h - U|$ and $E_U^n = \frac{\|U - U_{2-n}\|_{\overline{\Pi}^h}}{\|U - U_{2-(n+1)}\|_{\overline{\Pi}^h}}$, where U be the trace of the exact solution of the continuous problem $\overline{\Pi}^h$, and U_h be its approximate values.

CONCLUSION

The proposed method can be used to obtain the derivative of the solution of the 3D Laplace equation with mixed boundary conditions.

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